Combinatorial Nullstellensatz

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Institut Teknologi Bandung

Combinatorial Mathematics Research Group
Faculty of Mathematics and Natural Sciences
Institut Teknologi Bandung

COMBINATORICS TODAY Series #2

Title

COMBINATORIAL NULLSTELLENSATZ

Tuesday, August 10th, 2021
15:00-16:30 (Western Indonesia Time)

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Outline

- Alon's theorem
- List coloring
- Additive transversals
- Graph decompositions
- Antimagic labelings
- Counting
- Counting group colorings of K_5–minor–free graphs
- Epilog

Combinatorial Nullstellensatz

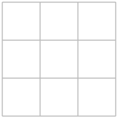
Theorem (Alon, 1999)

Let F be a field and $P \in F[x_1, \ldots, x_n]$ a polinomial on n variables over F with degree d.
Let $s_1, \ldots, s_n \geq 0$ with $s_1 + \cdots + s_n = d$. If the coefficient of the monomial

$$x_1^{s_1} \cdots x_n^{s_n},$$

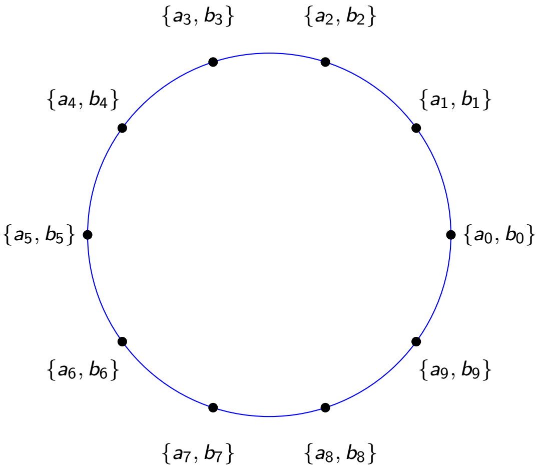
is nonzero, then P does not vanish in the grid $S_1 \times \cdots \times S_n$ for every collection of sets $S_1, \ldots, S_n \subset F$ such that $|S_i| < s_i$, $i = 1, \ldots, n$

$$P(x, y) = (x - 1)(y - 1) = xy - x - y + 1$$



A simple example

Two colors are assigned to every vertex of the cycle C_{10}. There is a proper coloring of the cycle where each vertex takes a color from its list.



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- Consider the polynomial
  $P(x_1, \ldots, x_{100}) = (x_1 - x_2)(x_2 - x_3) \cdots (x_9 - x_{10})(x_{10} - x_1)$

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- All exponents are one. By choosing S_i de two numbers assigned to vertex i, we can choose $(c_1, \ldots, c_{10}) \in S_1 \times \cdots \times S_{10}$ such that $P(c_1, \ldots, c_{10}) \neq 0$.

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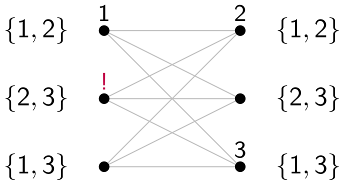
(What about a cycle of odd length?)

Original application: List coloring

List coloring

- $G = (V = \{1, 2, \dots n\}, E)$ be a graph.
- $L = \{L_i : i \in V\}$ family of lists, $L_i \subset \mathbb{N}$.
- $\chi_L : V \to \mathbb{N}$ list coloring: proper and $\chi_L(i) \in L_i$.
- $\chi_L(G)$, minimum k such that $\min_i |L_i| \geq k$ guarantees existence of a list coloring χ_L.



$\chi_L(K_{3,3}) > \chi(K_{3,3})$

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Consider the graph polinomial $P = \prod_{ij \in E, i<j}(x_i - x_j)$.

- $P(c_1, \dots, c_n) \neq 0 \Leftrightarrow \chi(i) = c_i$ proper coloring.
- P does not vanish on $L_1 \times \cdots \times L_n \Leftrightarrow$ there is a list coloring with lists $\{L_i : i \in V\}$.

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Theorem (Alon Tarsi)

Let D be an orientation of the graph G and $d^+(i)$ the outdegree of i in D . If

$$EE(D) \neq EO(D)$$

then there is a list coloring of G for every set of lists with $|L_i| > d^+(i)$.

$EE(D)$ and $EO(D)$ the number of Eulerian subgraphs of D with even and odd number of edges.

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- $P_G = \prod_{ij \in E, i<j}(x_i - x_j)$ has degree $d = \sum_i d_i$.
- By CombNull the result follows if $[x_1^{d_1}, \ldots, x_n^{d_n}]P_G \neq 0$.

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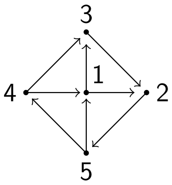
$$(x_2 - x_1)(x_3 - x_1)(x_4 - x_1)(x_5 - x_1)(x_5 - x_4)(x_4 - x_3)(x_3 - x_2)(x_5 - x_2)$$

$$[x_1^2 x_2 x_3 x_4^2 x_5^2]P_G = DE - DO = EE(D) - EE(O)$$



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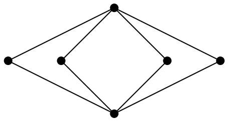
Theorem (Alon, Tarsi, 1992)

Let G be a bipartite planar graph. Then $\chi_L(G) = 3$.

- There is an orientation with out–degree at most 2.



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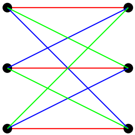
Theorem (Fleischner, Stiebitz, 1992)

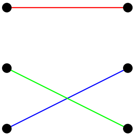
Let G be a graph on $3n$ vertices consisting of an edge–disjoint union of a Hamiltonian cycle and n paiwise distinct triangles. Then $\chi_L(G) = \chi(G) = 3$.

Additive transversals

Let G be equipped with an edge coloring. A rainbow matching is a matching with no two colors the same.



$\rightarrow$



$$\rightarrow \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

When $G = K_{n,n}$ and the coloring is given by sums, $c(i,j) = i + j \pmod{n}$ then

rainbow matching= additive transversal

Additive transversals

Let G be equipped with an edge coloring. A rainbow matching is a matching with no two colors the same.

Theorem (Alon, 2000)

Let $A, B \subset \mathbb{Z}/p\mathbb{Z}$ with $|A| = |B| = k$. There is an additive transversal of the Latin subsquare of the additive table of $\mathbb{Z}/p\mathbb{Z}$ induced by the two sets.

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- Let $B = \{b_1, \ldots, b_k\}$ and $P = \prod_{1 \leq i < j \leq k}(x_i + b_i - (x_j + b_j)) \prod_{1 \leq i < j \leq k}(x_i - x_j)$.

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- $deg(P) = k(k-1)$.
- $[x_1^{k-1} \ldots x_k^{k-1}]P = k! \neq 0$ (in $\mathbb{Z}/p\mathbb{Z}$ if $k < p$). One can replace $B \subset \mathbb{Z}/p\mathbb{Z}$ by B a sequence of k elements (with repetitions allowed).

Additive transversals

Let G be equipped with an edge coloring. A rainbow matching is a matching with no two colors the same.

Conjecture (Snevily, 1995)

Let $A, B \subset G$, G abelian group of odd order and $|A| = |B| = k$. There is a transversal of the Latin subsquare of the additive table of G induced by A and B.

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Theorem (Dasgupta, Károlyi, Serra, Szegedy, 2002)

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Theorem (Dasgupta, Károlyi, Serra, Szegedy, 2002)

Snevily conjecture holds for cyclic groups of odd order n.

- Since n is odd, $2^{\phi(n)} - 1 \equiv 0 \pmod{n}$. Choose $F = F_{2^{\phi(n)}}$.
- The multiplicative group of F has a cyclic subgroup of order n.
- $P = \prod_{1 \leq i < j \leq k} (x_j b_j - x_i b_i) \prod_{1 \leq i < j \leq k} (x_i - x_j)$.
- $[x_1^{k-1} \dots x_k^{k-1}] P = PermVand(b_1, \dots, b_k)$.
- In characteristic 2, $Perm = Det$.

Additive transversals

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Snevily conjecture was eventually proved for all abelian groups of odd order using sums of characters and an ingenious argument by Arsovsky (2011), at the time undergraduate student. (!)

Additive transversals

Let G be equipped with an edge coloring. A rainbow matching is a matching with no two colors the same.

Matchings with prescribed colors in $\mathbb{Z}/p\mathbb{Z}$.

Theorem (Lladó, Moragas, 2015)

Let $(m_1, \dots, m_k)$ be a sequence of elements in $\mathbb{Z}/p\mathbb{Z}$ with $k \leq (p-1)/2$. There are k disjoint pairs $\{a_1, b_1\}, \dots, \{a_k, b_k\}$ such that $a_i + b_i = m_i$ for $i = 1, \dots, k$.

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- $P = \prod_{1 \leq i < j \leq 2k}(x_i - x_j) \prod_{i=1}^{k}(1 - (x_i + x_{k+1} - m_i)^{p-1})$
- If $P(a_1, \ldots, a_k, b_1, \ldots, b_k) \neq 0$ we are done.
- Apply Alon's theorem.

Conjectures of Ringel and Graham–Haggkvist

Conjecture (Ringel, 1963)

Every tree with m edges decomposes K_{2m+1}.

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Theorem (LLadó, 2018)

Every random tree with m edges almost decomposes K_{2m+1} *almost surely.*

Proof by the polynomial method.

Conjectures of Ringel and Graham–Haggkvist

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Theorem (Montgomery, Pokrovskiy, Sudakov, 2020)

Ringel conjecture holds true for every large n.

Combinatorial proof.

Conjectures of Ringel and Graham–Haggkvist

Conjecture (Graham–Haggkvist, 1981)

Every tree with m edges decomposes $K_{m,m}$.

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Every tree with m edges decomposes $K_{m,m}$.

Theorem (Drmota, LLadó, 2017)

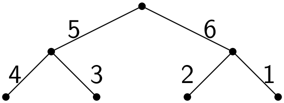
Every random tree with m edges decomposes $K_{m,m}$ almost surely.

Proof uses again the polynomial method.

Antimagic labelings

Antimagic labeling

An edge labeling of a graph is antimagic if the sums of labels of edges incident to vertices are pairwise distinct.



Conjecture (Harstfield-Ringel)

Every graph $G \neq K_2$ is antimagic.

- Stars, cycles, paths, complete graphs are antimagic (Harstfield-Ringel)
- Complete r–partite graphs are antimagic (Alon)
- Regular bipartite graphs are antimagic (Cranston)
- ...

Antimagic labelings

Theorem (LLadó, Miller, 2017)

Every graph with n vertices and $m > 1$ edges admits an antimagic labeling with $[2n+m-4]$ labels. A tree whose base tree has k vertices admits an antimagic labeling with $m+k$ labels.

- Associate edges with variables $x_1, \dots, x_m$.
- For every vertex define the variable $y_i = \sum_{e(v_i)} x_e$.
- The polynomial $V(x_1, \dots, x_m)V(y_1, \dots, y_n)$ (V Vandermonde polynomial) does not vanish at $(a_1, \dots, a_m)$ if and only if $f(e_i) = a_i$ is an antimagic labeling.
- Every monomial $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ has nonzero coefficient if $\alpha_i \leq (m-1) + 2(n-2)$.

Counting

How many points not vanishing in the grid?

Theorem (Alon, Füredi, 1993)

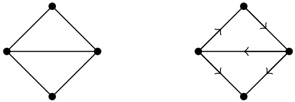
Let F be a field and $P \in F[x_1, \ldots, x_n]$ a polinomial on n variables over F with degree d.
If P does not vanish on the grid $S_1 \times S_n$ then there are at least

$$\min \left\{ \prod_{i=1}^{n} y_i : y_i \leq |S_i|, \sum_i y_i \leq \sum_i |S_i| - d \right\}.$$

Group colorings

Modular colorings

A graph G admits a modular n–coloring if there is some orientation of G such that for every labeling $l : \mathbb{Z}_n \to E(G)$ there is a coloring $c : V(G) \to \mathbb{Z}_n$ such that $c(c) - c(y) \neq l(x, y)$ for every directed edge (x, y).



A notion introduced by Jaeger, Linial, Payane and Tarsi as a dual of group connectivity

Group colorings

Theorem (Langhede, Thomassen, 2020)

Every planar graph on n vertices admits $2^{n/9}$ $\mathbb{Z}_5$–colorings.

Theorem (Bosek, Grytczuk, Gutowski, Serra, Zacaj, 2021)

Every K_5–minor graph on n vertices admits $5^{n/4}$ $\mathbb{Z}_5$–colorings.

Epilogue

- Alon's paper has (according to Google) 600 citations. A very good paper to read.
- There are several important additional applications in Finite Geometry, Algebra, Number Theory and more not discussed here.
- There are refinements, multiset versions, multiplicity of roots,...
- It is a useful tool which still deserves exploring new applications and refining its performance.