Diagonally Cyclic Latin Squares

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Inc. joint work with Jack Allsop and Aleš Drápal

Diagonally Cyclic Latin Squares

A *Latin square* is a square matrix in which each row and column is a permutation of the same set of symbols.

$$\begin{bmatrix} 0 & 2 & 5 & 1 & 6 & 4 & 3 \\ 4 & 1 & 3 & 6 & 2 & 0 & 5 \\ 6 & 5 & 2 & 4 & 0 & 3 & 1 \\ 2 & 0 & 6 & 3 & 5 & 1 & 4 \\ 5 & 3 & 1 & 0 & 4 & 6 & 2 \\ 3 & 6 & 4 & 2 & 1 & 5 & 0 \\ 1 & 4 & 0 & 5 & 3 & 2 & 6 \end{bmatrix}$$

A Latin square is *diagonally cyclic* if the symbols occur in cyclic order along each broken diagonal parallel to the main diagonal.

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The only problem is if $\theta(a) + b - a = \theta(b)$ for some a and b.

So we want $\theta(a) - a \neq \theta(b) - b$ for all a, b.

Orthomorphisms

An *orthomorphism* of an abelian group G is a permutation $\theta : G \mapsto G$ such that the map

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There is a DCLS with first row $[\theta(0), \theta(1), \ldots, \theta(n-1)]$ iff θ is an orthomorphism of $\mathbb{Z}_n$.

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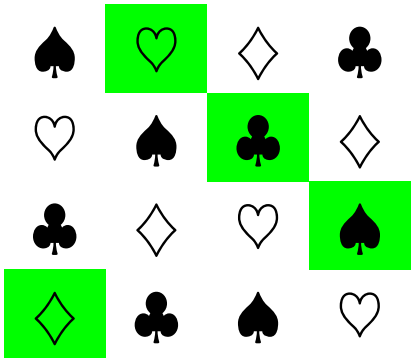
Their structure is not so visually appealing but the theory works the same.

Transversals

A *transversal* of a latin square is a set of entries which includes exactly one entry from each row and column and one of each symbol.



A	*K*	*J*	*Q*
Q	*J*	*K*	*A*
J	*Q*	*A*	*K*
K	*A*	*Q*	*J*

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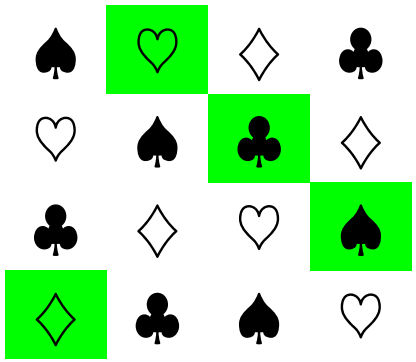

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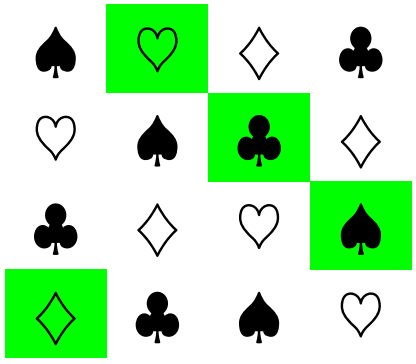

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Theorem: [Euler] The addition table for $\mathbb{Z}_n$ has a transversal iff n is odd.

A group has a transversal iff it has an orthomorphism:
Choose the triple $\big(x,\, \theta(x){-}x,\, \theta(x)\big)$ for each x.

Linear orthomorphisms

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Over $\mathbb{Z}_n$, we need $\gcd(\lambda, n) = 1 = \gcd(\lambda - 1, n)$.
Over $\mathbb{F}_q$, we need $\lambda \notin \{0, 1\}$.

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Orthogonal cyclotomic orthomorphisms are the "best" way to construct mutually orthogonal latin squares.

The 16 card trick

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$\spadesuit A$	$\heartsuit K$	$\diamondsuit J$	$\clubsuit Q$
$\heartsuit Q$	$\spadesuit J$	$\clubsuit K$	$\diamondsuit A$
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♠A	♡K	♢J	♣Q
♡Q	♠J	♣K	♢A
♣J	♢Q	♡A	♠K
♢K	♣A	♠Q	♡J

Each solution is the superposition of two latin squares

♠	♡	♢	♣
♡	♠	♣	♢
♣	♢	♡	♠
♢	♣	♠	♡

A	K	J	Q
Q	J	K	A
J	Q	A	K
K	A	Q	J

These squares are *orthogonal mates*.
When we overlay them each ordered pair of symbols occurs once.

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For prime p, Evans ['92] showed how to build a maximal set of

$$\begin{cases} (p-1)/2 & \text{if } p \equiv 1 \mod 4 \\ (p-3)/2 & \text{if } p \equiv 3 \mod 4 \end{cases}$$

orthogonal LS using a quadratic orthomorphism and all the linear orthomorphisms orthogonal to it**.

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Conjecture: Asymptotically almost all orthomorphisms have degree $q-3$.

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With Shallue (2013), we found all orthomorphism polynomials of degree $\leqslant 6$.

Subsquares

A *subsquare* is a submatrix that is itself a Latin square. For example here are a subsquare of order 3 and another of order 2.

$$\begin{bmatrix} 0 & 2 & 5 & 1 & 6 & 4 & 3 \\ 4 & 1 & 3 & 6 & 2 & 0 & 5 \\ 6 & 5 & 2 & 4 & 0 & 3 & 1 \\ 2 & 0 & 6 & 3 & 5 & 1 & 4 \\ 5 & 3 & 1 & 0 & 4 & 6 & 2 \\ 3 & 6 & 4 & 2 & 1 & 5 & 0 \\ 1 & 4 & 0 & 5 & 3 & 2 & 6 \end{bmatrix}$$

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Theorem: [Maenhaut/Webb/W.'07] For all odd orders there exist Latin squares with no proper subsquares.

(The analogous problem for even orders is harder, and isn't completely resolved).

Row Cycles

2 rows of a LS define a permutation, which decomposes into cycles.

	0	2	5	1	6	4	3
	4	1	3	6	2	0	5
	6	5	2	4	0	3	1
$\rightarrow$	2	0	6	3	5	1	4
$\rightarrow$	5	3	1	0	4	6	2
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Similarly, we have column-Hamiltonian and symbol-Hamiltonian.

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-Hamiltonian LS correspond to perfect 1-factorisations of $K_{n,n}$. They also have no proper subsquares.

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A quasigroup is *maximally non-associative* if $(xy)z = x(yz)$ only when $x = y = z$. Such quasigroups apparently have some application in cryptography.

Joint work with Aleš Drápal

Maximally non-associative quasigroups do not exist for orders $1, \ldots, 8$.

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Theorem: A maximally nonassociative quasigroup of order n exists for all $n \geqslant 9$, with the possible exception of $n \in \{11, 12, 15, 40, 42, 44, 56, 66, 77, 88, 90, 110\}$ and orders of the form $n = 2p_1$ or $n = 2p_1p_2$ for odd primes p_1, p_2 with $p_1 \leqslant p_2 < 2p_1$.

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Theorem: For an odd prime power q the asymptotic proportion of quadratic orthomorphisms which produce maximally nonassociative quasigroups is

$$\begin{cases} \frac{953}{2^{15}} \approx 0.02908 & \text{for } q \equiv 1 \bmod 4, \\ \frac{825}{2^{16}} \approx 0.01259 & \text{for } q \equiv 3 \bmod 4. \end{cases}$$

That's all. Any questions?